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Kinetic-energy mass, momentum mass, and drift mass in steady irrotational subsonic flows

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Irrotational flows caused by a body moving with a constant velocity in an unbounded homentropic compressible fluid at rest at infinity are considered. Provided the (steady) flow relative to the body is everywhere subsonic, it is shown that the momentum mass is always equal to the drift mass, and the kinetic-energy mass is equal to the drift mass under certain conditions.

1. Introduction

In the derivation of the Darwin theorem (Darwin 1953) presented in a previous paper (Yih 1985), I calculated the drift mass by first integrating from $x = -\infty$ to $x = \infty$ in the steady flow relative to the moving body, and then integrating across the streamlines, and showed how Darwin's theorem on the equality of drift mass and added mass can be very simply derived. In this paper I shall consider the drift mass m_d caused by the translation of a body in a compressible fluid, and relate it to the kinetic-energy mass m_k of the fluid, provided that the flow relative to the body is steady, irrotational, and everywhere subsonic.

By adopting the idea of partial drift volume initiated by Eames, Belcher & Hunt (1994), a weak form of the equality of m_d and m_k is obtained which contains, on letting the domain of integration become infinite, the strong form of Darwin's theorem for compressible fluids, i.e. the equality of the drift mass and the kinetic-energy mass. Both the two-dimensional and three-dimensional cases are treated.

I take this opportunity to introduce the idea of the momentum mass and to show its equality to the drift mass, thus justifying Darwin's *definition* of the drift mass by the momentum integral. The intervention of the momentum mass largely relieves us of the burden of having to explain the puzzling equality of m_k, which has a kinetic significance, and m_d, which is purely kinematic in origin.

The resistance encountered by a body accelerating in a compressible fluid is not equal to m_k times the acceleration, and can be considerably larger than that product. That is why I have not called m_k the added mass. This matter will be discussed in the last section of this paper.

2. Analyses

We consider the steady subsonic irrotational flow of a compressible fluid relative to a body moving from $x = \infty$ to $x = -\infty$. The origin of Cartesian coordinates (x, y, z) is taken inside the body, most conveniently at its centre of gravity. The velocity components in the directions of increasing values of the coordinates in this steady flow are denoted by (u, v, w), respectively, and those in the unsteady flow caused by the moving body are denoted by (u', v', w'), so that

$$u = 1+u', \quad v = v', \quad w = w',$$

if the speed of the body is taken to be unity for convenience. The velocity potential and the stream functions for the steady flow will be denoted by (ϕ, ψ, χ) for three-dimensional flows and (ϕ, ψ) for two-dimensional flows. The velocity potential for the unsteady flow is denoted by ϕ'. We consider a plane initially at $x = -\infty$ that moves kinematically with velocity 1 to the right, and mark (by dye, say) the fluid particles initially at the plane $x = -\infty$. The volume between the marked particles and the kinematic plane is the drift volume, and the final drift volume, when the kinematic plane has reached $x = \infty$, is the volume of interest. This definition of the drift volume is in agreement with Darwin's original, somewhat different, definition.

For the steady-flow relative to the body, we have, to start with,

$$(u, v, w) = \operatorname{grad} \phi, \tag{1}$$

which is a consequence of irrotationality. The equation of continuity is

$$(\rho u)_x + (\rho v)_y + (\rho w)_z = 0, \tag{2}$$

where the subscripts indicate partial differentiation. This equation allows one to write

$$(\rho u, \rho v, \rho w) = \rho_0 \operatorname{grad} \psi \times \operatorname{grad} \chi, \tag{3}$$

where ρ_0 is the density at infinity, and ψ and χ are two stream functions. For two-dimensional flows $(w = 0)$ we have

$$(\rho u, \rho v) = \rho_0(\psi_y, -\psi_x). \tag{4}$$

From (1) and (2) it follows that

$$\nabla^2 \phi = -\frac{1}{\rho}\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\rho. \tag{5}$$

The right-hand side of (5) can be regarded as distributed exterior sources and sinks in an incompressible flow. But the sum S of these sources is

$$\begin{aligned} S &= -\iiint \frac{1}{\rho}\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\rho \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= -\iiint \frac{1}{\rho}\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\rho\left(\frac{\mathrm{d}\phi\,\partial\psi\,\mathrm{d}\chi}{J}\right) \\ &= -\iiint \frac{\rho_0}{\rho^2}\left(\frac{u}{q^2}\frac{\partial}{\partial x} + \frac{v}{q^2}\frac{\partial}{\partial y} + \frac{w}{q^2}\frac{\partial}{\partial z}\right)\rho \,\mathrm{d}\phi\,\mathrm{d}\psi\,\mathrm{d}\chi, \end{aligned} \tag{6}$$

where J is the Jacobin defined by

$$J = \frac{\partial(\phi, \psi, \chi)}{\partial(x, y, z)} = \frac{\rho}{\rho_0}q^2, \tag{7}$$

on account of (1) and (3), with

$$q^2 = u^2 + v^2 + w^2. \tag{8}$$

But

$$\frac{1}{q^2}(u, v, w) = (x_\phi, y_\phi, z_\phi), \tag{9}$$

so that (6) can be written as

$$S = \iiint \frac{\partial}{\partial \phi}\frac{\rho_0}{\rho}\,\mathrm{d}\phi\,\mathrm{d}\psi\,\mathrm{d}\chi = 0. \tag{10}$$

Hence there is no net virtual source arising from the right-hand side of (5), as indeed is to be expected.

For any given body, the flow is determined by the well-known Rayleigh–Jansen method, in which the right-hand side of (5) is taken into account, and the solution will contain an interior doublet, which, together with the uniform flow, gives the flow at infinity. One might be concerned with the question of how far the exterior virtual sources are distributed. In principle they go all the way to infinity. But a quick calculation of the right-hand side of (5), upon use of the Bernoulli equation connecting the density ρ and the velocity, shows that the strengths of these sources drop off so fast as not to affect at all the calculations that follow.

For the two-dimensional case, consider fluid drift in the domain D bounded externally by

$$x = -x_0, \quad x = x_0, \quad \psi = -\psi_B, \quad \psi = \psi_B, \tag{11}$$

and internally by the body under consideration. In the (ϕ, ψ)-space, (11) has the form

$$\phi = \phi(-x_0, y), \quad \phi = \phi(x_0, y), \quad \psi = -\psi_B, \quad \psi = \psi_B. \tag{12}$$

The crucial equation is

$$I = I_1 - I_2, \tag{13}$$

where

$$I = \iint \rho\,[(u-1)^2 + v^2]\,\mathrm{d}x\,\mathrm{d}y, \tag{14}$$

$$I_1 = \iint \rho(1-u)\,\mathrm{d}x\,\mathrm{d}y, \tag{15}$$

$$I_2 = \iint \rho(u-q^2)\,\mathrm{d}x\,\mathrm{d}y, \tag{16}$$

$$q^2 = u^2 + v^2,$$

and all integrations are carried over D. The integral I represents the kinetic-energy mass m_k, and I_1 represents the drift mass m_d, as explained in Yih (1985). By virtue of (1) (for the two-dimensional case) and (4),

$$I_1 = \rho_0 \iint \left(\frac{1-u}{q^2}\right) \mathrm{d}\phi\,\mathrm{d}\psi, \quad I_2 = \rho_0 \iint \left(\frac{u}{q^2} - 1\right) \mathrm{d}\phi\,\mathrm{d}\psi = \rho_0 \iint \left(\frac{\partial x}{\partial \phi} - 1\right) \mathrm{d}\phi\,\mathrm{d}\psi. \tag{17}$$

The integral I_1 defined by (15) is Darwin's definition of the drift mass when x_0 and ψ_B become infinite. It is, however, *prima facie* a momentum mass. To show how it is related to the drift mass in D, we again use the idea

$$\frac{1}{q^2}\,\mathrm{d}\phi = \mathrm{d}t$$

in I_1 given by (17). For the domain D, the inner integral $\int q^{-2}\,\mathrm{d}\phi$ is the time required for any particle to go from $-x_0$ to x_0 along a streamline. The limits of integration are $\phi(-x_0, \psi)$ and $\phi(x_0, \psi)$. The other inner integral in I_1 is

$$\int \frac{u}{q^2}\,\mathrm{d}\phi = \int \frac{\partial x}{\partial \phi}\,\mathrm{d}\phi = 2x_0.$$

Hence I_1 is the integral, with respect to $\rho_0 \psi$, of the difference of the time required for a particle to go from $-x_0$ to x_0 along a streamline and the time required for a particle at $|y| = \infty$ to do the same, and hence is the drift mass for the domain D if we assume that u near $x = x_0$ is already 1. It is not yet. But, as (19) will show,

$$1-u = u' = O(x_0^{-2}).$$

Thus, for domain D,

$$m_m = m_d + O(x_0^{-2}). \tag{18a}$$

As x_0 and ψ_B become infinite, then for the entire fluid

$$m_m = m_d, \tag{18b}$$

which justifies Darwin's definition of the drift mass. Note that the idea of the excess required time of travel has been used to equate a mass of dynamical nature to one of kinematical origin. The same idea can be used for the same purpose in the three-dimensional case.

For large x and y, the presence of the body can be represented by a virtual doublet of strength a^2, where a is an equivalent radius. Then for large x and y,

$$\phi = x + \frac{a^2(lx+my)}{r^2}, \quad \psi = y - \frac{a^2(ly-mx)}{r^2}, \quad r^2 = x^2+y^2, \tag{19}$$

in which $(-l, -m)$ are the direction cosines of the axis of the doublet. Then, according to (17),

$$I_2 = \rho_0 \int [(x-\phi)_{x_0} - (x-\phi)_{-x_0}]\,\mathrm{d}\psi = -2a^2\rho_0 \int \frac{lx_0}{x_0^2+\psi^2}\,\mathrm{d}\psi + O(x_0^{-2}),$$

where the limits of integration are $-\psi_B$, and ψ_B, or

$$I_2 = -4a^2\rho_0\, l \arctan\frac{\psi_B}{x_0} + O(x_0^{-2}).$$

Upon letting x_0 and ψ_B approach infinity, and keeping their ratio constant, we have, from (13),

$$m_k = m_m + 4\rho_0 a^2 l \arctan\frac{\psi_B}{x_0}, \tag{20}$$

where m_k and m_m are, respectively, the kinetic-energy mass and the momentum mass.

If the ratio ψ_B/x_0 approaches zero as $x_0 \rightarrow \infty$ (Eames *et al.* 1994), which amounts to integrating with respect to x from $-\infty$ to ∞ before integrating with respect to y, as done in Yih (1985), we have, upon using (18) and (20), the strong form of Darwin's theorem:

$$m_k = m_d.$$

If that ratio is infinite, which amounts to integrating with respect to y from $-\infty$ to ∞ first,

$$m_k = m_m + 4\pi\rho_0 a^2 l.$$

This shows that m_m and hence m_d may be negative, i.e. the drift may be really a reflux. This is understandable because, for infinite ψ_B/x_0, in most of D (before x_0 goes to infinity) the speed exceeds 1. It is below 1 only in a relatively small bounded region before and after the body. We note in passing that the integral I_2 can be written as

$$\rho_0 \iint \left(\frac{\partial y}{\partial \psi} - 1\right) \mathrm{d}\psi\,\mathrm{d}\phi$$

and the integration with respect to ψ performed first in the domain D. The result is the same provided proper attention is given to the discontinuity of y at $\psi = 0$ at the surface of the body. When the domain is finite, the order of integration should not, and does not, matter.

The last term in (20), when not zero, destroys the equality of m_k and m_d. But Darwin's theorem still has a physical significance, because that term comes from the integral I_2 with a very small integrand spread over a large interval in y (or ψ) as x_0 and ψ_B both become large. Hence if the integration with respect to ψ is carried out, with

$$\psi_B = Na, \quad N \gg 1, \tag{21}$$

and if

$$x_0 = N^3 a, \tag{22}$$

we have from (20), (18) and (22), as N becomes very large, the weak form of Darwin's theorem:

$$m_k = m_d + O\left(\frac{1}{N^2}\right). \tag{23}$$

In (23), m_k is the kinetic-energy mass for the *entire* fluid domain, but m_d is for the domain D of integration. Note that the contribution to m_k from the region outside D is $O(N^{-2})$, as can be easily demonstrated by using (19), (21), and (22). Indeed, that is why we have chosen (22). To make the power (of N) larger than 3 would gain no accuracy; to make it less than 3 would lose sharpness. So Darwin's theorem is reborn in a weaker form – weaker mathematically but physically significant and useful to the experimenter. Darwin himself recognized the physical significance of his result, in spite of its indeterminacy. What I have done here is to put his theorem into a weak but more precise form. The explanation of the choice of (22) and of the last term in (23) shows that (21), (22), and (23) say rather more than the results of Eames *et al.* (1994).

The three-dimensional case can be similarly treated. Writing now

$$R^2 = x^2 + r^2, \quad r^2 = y^2 + z^2,$$

we note that for large R

$$\phi = x + \frac{a^3(lx + my + nz)}{R^3}, \tag{24}$$

in which $(-l, -m, -n)$ are the direction cosines of the axis of the virtual doublet. The key equation is still

$$I = I_1 - I_2, \tag{25}$$

where now

$$I = \iiint \rho[(u-1)^2 + v^2 + w^2]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{26}$$

$$I_1 = \iiint \rho(1-u)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{27}$$

$$I_2 = \iiint \rho(u - q^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{28}$$

$$q^2 = u^2 + v^2 + w^2.$$

By virtue of (1) and (3),

$$I_1 = \iiint \rho_0\left(\frac{1-u}{q^2}\right)\mathrm{d}\phi\,\mathrm{d}\psi\,\mathrm{d}\chi, \tag{29a}$$

$$I_2 = \iiint \rho_0\left(\frac{u}{q^2} - 1\right)\mathrm{d}\phi\,\mathrm{d}\psi\,\mathrm{d}\chi = \rho_0 \iiint \left(\frac{\partial x}{\partial \phi} - 1\right)\mathrm{d}\phi\,\mathrm{d}\psi\,\mathrm{d}\chi. \tag{29b}$$

The equation for the three-dimensional case corresponding to (18a) for the two-dimensional case is

$$m_m = m_d + O(x_0^{-3}) \tag{30a}$$

for domain D, which for infinite x_0 and r_0 becomes, for the entire fluid,

$$m_m = m_d, \tag{30b}$$

again justifying Darwin's definition of the drift mass.

We shall choose ψ to become ultimately the Stokes stream function at large r. The domain D of integration is bounded externally by

$$\psi = \psi_0, \quad x = -x_0, \quad x = x_0 \tag{31}$$

and internally by the body. At $x = \pm x_0$ and for large x_0, we can write

$$\mathrm{d}\psi\,\mathrm{d}\chi = r\,\mathrm{d}r\,\mathrm{d}\theta, \tag{32}$$

where r and θ are polar coordinates in the (y, z)-plane. The error is insignificant, and vanishes as $x_0 \rightarrow \infty$.

Using (24) and (30) and ignoring insignificant terms that vanish as $x_0 \rightarrow \infty$, we have

$$I_2 = 4\pi a^3 l\rho_0\left[\frac{x_0}{(x_0^2+r_0^2)^{1/2}} - 1\right]. \tag{33}$$

Upon keeping the ratio r_0/x_0 constant and letting $x_0 \rightarrow \infty$, (25) becomes

$$m_k = m_m + 4\pi\alpha^3 l\rho_0\left[1 - \frac{x_0}{(x_0^2+r_0^2)^{1/2}}\right], \tag{34}$$

and all that has been said on the two-dimensional case can be repeated here. In particular, if $r_0/x_0 \rightarrow 0$ as r_0 and $x_0 \rightarrow \infty$ (Eames *et al.* 1994), as was in effect satisfied in Yih (1985), we have from (30) and (34) the strong form of Darwin's theorem:

$$m_k = m_d. \tag{35}$$

By integrating from zero to r_0, with

$$r_0 = Na, \quad N \gg 1, \tag{36}$$

and by taking

$$x_0 = N^{5/2}a, \tag{37}$$

we have, from (30) and (34),

$$m_k = m_d + O(N^{-3}), \tag{38}$$

where m_k is the kinetic-energy mass for the *entire* fluid domain, but m_d is for the domain of integration D. It can be shown that the contribution to m_k from outside D is $O(N^{-3})$, and that is why (37) is chosen for maximum sharpness of (38) without making the exponent of N in (37) unnecessarily large. Thus Darwin's theorem is reborn again in a weaker form, now for the three-dimensional case. Again, (36) to (38) say rather more than the Eames *et al.* paper.

We have refrained from calculating the m_k and m_d for any specific shape of the body. Once the velocity potential ϕ for a flow is given, this calculation is straightforward though tedious. For a circular cylinder ϕ is given in Van Dyke (1975, p. 16), and for a sphere it is given in Lighthill (1954, equations (2–9), p. 355). We call attention to the fact that in these results, as well as in equation (4.5) of an early article by Batchelor (1945), the terms containing r^{-1} or R^{-2}, in the two-dimensional and three-dimensional cases, respectively, are

$$a_1 r^{-1}\cos\theta + a_3 r^{-1}\cos 3\theta, \tag{39}$$

and

$$b_1 R^{-2}\cos\theta + b_3 R^{-2}\cos 3\theta, \tag{40}$$

where θ is either a polar coordinate or a spherical coordinate. In (39) the first term represents a doublet. The second term is equal to

$$a_3(r^{-1}\cos\theta - 4r^{-1}\cos\theta\sin^2\theta). \tag{41}$$

In (41), the first term is a doublet that can be combined with the a_1-term in (39), and the second term, for $\psi_B/x_0 = O(N^{-2})$, would give a term of $O(N^{-6})$ on the right-hand side of (20). The same can be said of the three-dimensional case. (The a^2 in (19) and a^3 in (24) are supposed to be the magnitude of the *sum* of the doublets.) Thus the weak form of the Darwin theorem, and *a fortiori* its strong form, are unaffected by terms like $r^{-1}\cos 3\theta$ and $R^{-2}\cos 3\theta$. These arise from the compressibility of the fluid, and can be expected to be present for arbitrarily shaped bodies, not just for circular cylinders or spheres. Indeed, Batchelor (1945) did not deal with any specific body shape at all.

3. Relation between the relative internal energy and the kinetic energy

The assumption of unbounded fluid space necessarily excludes the consideration of gravity effects. If gravity effects are neglected, the Bernoulli equation is

$$\frac{\gamma}{\gamma-1}\left(\frac{p}{\rho}-\frac{p_0}{\rho_0}\right)+\tfrac{1}{2}(q^2-1) = 0, \tag{42}$$

where p is the pressure, p_0 is p at infinity, and γ is the ratio of c_p, the specific heat at constant pressure, and c_v, the specific heat at constant volume. Since for ideal gases

$$R\rho T = p, \quad R = c_p - c_v,$$

where T is the absolute temperature, we have

$$\gamma c_v(T-T_0)-\tfrac{1}{2}(1-q^2) = 0, \tag{43}$$

where T_0 is T at infinity. Assuming c_v to be constant, we can define a relative internal energy per unit mass by

$$e_r = c_v(T-T_0), \tag{44}$$

multiply (43) by ρ, and integrate over the domain D defined by (21) and (22) in the plane case and by (36) and (37) in the three-dimensional case. The result is, when we recall the definition of I_1 by (15) or (27) and use E_r for the integral of e_r,

$$\gamma E_r = \tfrac{1}{2}I_1+\tfrac{1}{2}I_2 \tag{45}$$

for domain D, or, on account of (23) or (38),

$$\gamma E_r = \tfrac{1}{2}I \tag{46}$$

for the entire fluid, when N becomes infinite, with I defined by (14) or (26). Since the velocity of the body is taken to be unity (or is used as the velocity scale), $I/2$ represents the kinetic energy, in the laboratory frame, of the entire fluid, while E_r is its relative internal energy.

Thus the total energy of the entire fluid is

$$E_r+\tfrac{1}{2}I = \frac{\gamma+1}{2\gamma}\,I = \frac{\gamma+1}{\gamma}\,(\text{kinetic energy}). \tag{47}$$

Since the kinetic theory of gases has shown that T, and therefore $c_v\,T$, is proportional to the sum of kinetic energies of the molecules of a gas macrospically at rest, (47) is not entirely surprising. But it does show that m_k is not only the kinetic-energy mass, but is the energy mass as well, in the sense that the total energy of the entire fluid is

$$\frac{\gamma+1}{2\gamma}m_k\,U^2,$$

if U denotes the (dimensional) speed of the moving body. This is of some interest in its own right.

4. Concluding remark

I conclude this paper with an explanation of why I have refrained from identifying m_k or m_m with an added mass (denoted by m_a in my previous paper on fluids with constant density). For a fluid with constant density, $m_k = m_m = m_a$, and the added mass has the dynamical significance that when a body is accelerated in translation through the fluid it encounters a resistance equal to m_a times the acceleration. This is how the term 'added mass' arose in the literature. If a body accelerates in a compressible fluid, the resistance it meets is not equal to m_k (or m_m) times the acceleration, and may indeed be considerably larger. It is not constant with time even if the acceleration is constant. All this is obvious, since as the velocity of the body varies the flow is not determined by its instantaneous velocity alone but is dependent on the history of its motion, and since both energy and momentum may radiate toward infinity by means of sound waves – most obviously when the velocity of the body has an oscillatory component. So it would be misleading to call m_k (or m_m) the added mass.

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